MATH 112A Review: Laplace and Polar Coordinates

Facts to Know:

Check out the Prerequiste Materials for Math 2E on partial derivatives (video 3):

`https://www.math.uci.edu/~prerequisite-videos/2e.html`

(2 D) Let $f : \mathbb{R}^2 \to \mathbb{R}$. The Laplace operator is

$$\Delta f = f_{xx} + f_{yy}$$

(polar coordinates) Let $x = r\cos\theta$ and $y = r\sin\theta$. Let $g(r,\theta) = f(r\cos\theta, r\sin\theta)$. Then,

$$\Delta f(x,y)|_{(r\cos\theta, r\sin\theta)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} g$$

(3 D) Let $f : \mathbb{R}^3 \to \mathbb{R}$. The Laplace operator is

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

Examples:

1. Let $f(x,y) = xy^2$. What is Δf?

$f_x = y^2$ $\quad f_{xx} = 0$

$f_y = 2xy$ $\quad f_{yy} = 2x$

$\Delta f = 0 + 2x$

2. Let $f(x,y) = \sqrt{x^2+y^2}$. What is Δf in polar coordinates?

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= r\sqrt{\cos^2\theta + \sin^2\theta}$$
$$= r$$

$g_r = 1$

$g_\theta = 0$ $\quad g_{\theta\theta} = 0$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot g_r\right) + 0$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r) = \frac{1}{r}\cdot 1 = \frac{1}{r}.$$